Some perturbation solutions in laminar boundary-layer theory

Part 1. The momentum equation

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The velocity fields associated with a variety of flows which may be described by perturbations of the Blasius solution are considered. These are flows which, for example, because of localized mass transfer, involve the initial-value problem of boundary-layer theory, or which involve a variable ratio of the viscosity-density product, or finally which involve mass transfer. The perturbation solutions are presented so that in accord with the usual linearization procedures further applications for the determination of first-order effects can be readily made. In addition, each of these perturbations involves a common differential operator whose eigenfunctions form a complete orthogonal set. Accordingly, a procedure for systematically improving each perturbation solution to obtain higher-order effects by quadrature is presented. The results of applications in several cases are given and are compared to more accurate solutions where available.

1. Introduction

Despite the extensive advances in laminar boundary-layer theory there remain many problems which are only solvable either by numerical solution of the exact equations or by approximate methods, e.g. by the Kármán–Pohlhausen method. The former, which are identified currently with Flugge-Lotz & Baxter (1956, 1957), Kramer & Lieberstein (1959), Howe (1959), and Smith & Clutter (1961), will become increasingly more commonplace and will be applied to more complex problems. Similar comment applies to the Pallone–Dorodnitzn method (1961) which involves application of a strip method of solution. At the other limit of the accuracy-complexity spectrum are conventional integral methods which provide ready answers to many problems but which have well-known limitations with respect to detailed information and effects.

An approximation technique is presented here which is based on linearization about the Blasius solution and which provides simple and in principle exact solutions to some problems not tractable by conventional integral methods. In addition, it may be useful in conjunction with numerical solutions in regions where the Blasius solution is being approached asymptotically. Since it is based on linearization, it permits first-order effects of several co-existing perturbations to be readily established. In addition, because of the nature of the solutions obtained, systematic consideration of higher-order effects by quadrature is possible.

Expansion and perturbation techniques have been employed extensively in boundary-layer theory in the past. Schlicting (1955) and Hayes & Probstein (1959) provide ready reference thereto. The work of Glauert (1956), of Bloom & Steiger (1961), and of Ferri (1960) is closely related to the present study in so far as first-order effects are concerned.

In the following section the perturbation solutions for three types of velocity disturbances are obtained; these pertain to disturbances arising from the initial profile, from a variable ratio of the product of density and viscosity, and from the wall boundary condition. The solution corresponding to the first type of disturbance is shown to be given in terms of eigenfunctions which form a complete, orthogonal set and which relate to a differential operator common to all perturbations about the Blasius solution. Consequently, it is possible to obtain systematically higher-order solutions for each type of disturbance. In a subsequent section several applications are given along with comparisons to more accurate calculations where available.

2. Analysis

The momentum equation for a laminar boundary layer with a uniform external stream can be written in terms of the Levy–Lees variables η and $\tilde{s}$ as [cf. Lees (1956) and Hayes & Probstein (1959)]

$$(Cf_{\eta\eta})_\eta + ff_{\eta\eta} = 2\tilde{s}(f_\eta f_{\tilde{s}\eta} - f_{\eta\eta} f_{\tilde{s}}), \tag{2.1}$$

where

$$\eta = \rho_e u_e r^j (2\tilde{s})^{-\frac{1}{2}} \int_0^y (\rho/\rho_e)\, dy,$$

$$\tilde{s} = \int_0^x \rho_e \mu_e u_e r^{2j}\, dx.$$

Consider now a series of flows which are close to those described by the Blasius solution, namely by

$$f_0''' + f_0 f_0'' = 0, \tag{2.2}$$

subject to the boundary conditions

$$f_0(0) = f_0'(0) = 0; \quad f_0'(\infty) = 1.$$

Such a flow, for example, would be described as follows. Let

$$f_\eta(\tilde{s}, \eta) \simeq f_0'(\eta) + f_{1,1\eta}(\tilde{s}, \eta) + f_{1,2\eta}(\tilde{s}, \eta) + \ldots, \tag{2.3}$$

subject to the conditions

$$\left.\begin{aligned} f(\tilde{s}, 0) = f_\eta(\tilde{s}, 0) = 0, \quad f_\eta(\tilde{s}, \infty) = 1, \\ f_\eta(\tilde{s}_i, \eta) = F_0(\eta) \quad (\tilde{s}_i > 0), \end{aligned}\right\} \tag{2.4}$$

where $F_0(\eta)$ is compatible with (2.3) and where $f_{1,i+1} \ll f_{1,i}$. Thus, an initial profile deviating slightly from a Blasius profile is being considered.

The equation for $f_{1,1}$ is obtained by linearizing (2.1) about f_0 and is

$$f_{1,1\eta\eta\eta} + f_0 f_{1,1\eta\eta} + f_0'' f_{1,1} - 2\tilde{s}(f_0' f_{1,1\tilde{s}\eta} - f_0'' f_{1,1\tilde{s}}) = 0, \tag{2.5}$$

subject to the conditions

$$f_{1,1}(\tilde{s},0) = f_{1,1\eta}(\tilde{s},0) = 0, \quad f_{1,f1\eta}(\tilde{s},\infty) = 0,$$

and

$$f_{1,1\eta}(\tilde{s}_i,\eta) = F_0(\eta) - f_0'(\eta) \equiv \tilde{F}_0(\eta).$$

Separation of variables so that $f_{1,1}(\tilde{s},\eta) = N_1(\eta)\, S_1(\tilde{s})$ yields

$$S_1 \sim \tilde{s}^{-\frac{1}{2}\lambda_1}, \tag{2.6}$$

and

$$N_1''' + f_0 N_1'' + \lambda_1 f_0' N_1' + (1-\lambda_1) f_0'' N_1 = 0. \tag{2.7}$$

The conditions on N_1 are homogeneous: i.e. $N_1(0) = N_1'(\infty) = 0$; thus a typical eigenvalue problem arises. With no loss in generality $N_1''(0)$ can be set equal to unity so that with a value of λ_1 selected, a numerical integration of (2.7) proceeding from $\eta = 0$ can be carried out.

The asymptotic approximation for N_1' valid as $\eta \to \infty$, $|1-\lambda_1|\,(\eta - k)^{-2} \ll 1$, is shown in the Appendix to be

$$N_1' \sim \alpha_1(\eta-\kappa)^{-(1-\lambda_1)} \exp[-(\eta-\kappa)^2/2] + \alpha_2(\eta-\kappa)^{-\lambda_1} + \alpha_3 \exp[-(\eta-\kappa)^2/2]. \tag{2.8}$$

Inspection of (2.8) indicates that $N_1' \to 0$ as $\eta \to \infty$ for all positive λ_1; thus, a continuous spectrum of eigenvalues exists and numerical integration from $\eta = 0$ will result in $N_1'(\infty) = 0$ for any positive λ_1. However, if N_1' is required to exhibit exponential decay as $\eta \to \infty$, i.e. if $\alpha_2 = 0$, then only discrete values of λ_1 are acceptable. This requirement is employed herein along with a numerical procedure to determine the eigenvalues and related eigenfunctions. The details of the calculations are given in the Appendix.

Several comments on exponential behaviour of N_1' may be of interest. The asymptotic solution for (2.7) is closely related to the asymptotic solution of the momentum equation for similar flows considered by Cohen & Reshotko (1956). Indeed, their pressure gradient or similarity parameter β is equal to $-\frac{1}{2}\lambda_1$; for $\beta < 0$ they point out that non-unique solutions prevail unless exponential decay with $\eta \to \infty$ is required. In order to remove the non-uniqueness, they were able to invoke the physical argument valid for $\beta \geqslant -\frac{1}{2}$, i.e. for $\lambda_1 \leqslant 1$, namely that the displacement thickness becomes infinite unless $\alpha_2 = 0$. Since separation occurs for values of β satisfying this inequality in the cases they considered, this argument provides uniqueness therein but is inapplicable for $\lambda_1 > 1$. Similar considerations regarding the proper behaviour of solutions to the boundary-layer equations as $\eta \to \infty$ have been made by Goldstein (1956), Imai (1957) and Stewartson (1957); all argue that only exponential decay into the external potential flow is acceptable. Apparently these arguments are subject to considerable discussion. However, it is noted that the asymptotic behaviour of interest herein concerns a perturbation about a Blasius function which decays exponentially to unity; thus, in this case an exponential decay for the perturbations is required. Finally, note that the disturbing function $\tilde{F}_0(\eta)$ is restricted by these considerations to exponential decay as $\eta \to \infty$.

The solution for f_1 is thus given by

$$f_1 = \sum_{k=1}^{\infty} A_{1,k} \left(\frac{\tilde{s}}{\tilde{s}_i}\right)^{-\frac{1}{2}\lambda_{1,k}} N_{1,k}(\eta), \tag{2.9}$$

where the constants $A_{1,k}$ are determined by the condition

$$\tilde{F}_0(\eta) \simeq \sum_{k=1}^{\infty} A_{1,k} N_{1,k}(\eta). \tag{2.10}$$

It is of interest now to consider some properties of the $N_{1,k}$ functions; (2.7) can be reduced to a second-order linear equation of standard form by the substitutions $N_{1,k} = C_{1,k}(\eta) f_0'$, $H_{1,k} = C_{1,k}'$; then

$$[(f_1'^3/f_0'') H_{1,k}']' + [(\lambda_{1,k} f_0'^4/f_0'') - f_0 f_0'^2] H_{1,k} = 0, \tag{2.11}$$

which is an equation in Sturm–Liouville form. Now in the usual fashion consider two values of the index k, say m and n, and the two corresponding equations implied by (2.11). Then cross-multiplication by $H_{1,k}\, d\eta$, integration from 0 to ∞ and subtraction yields

$$\int_0^\infty (f_0'^4/f_0'') H_{1,m} H_{1,n}\, d\eta = C_n \delta_{mn}, \tag{2.12}$$

provided the $H_{1,k}$ functions decay exponentially as $\eta \to \infty$ at least as fast as $\exp[-(\eta-\kappa)^2/4]$. In terms of the $N_{1,k}$ functions, (2.12) implies that

$$\int_0^\infty (f_0'^4/f_0'') (N_{1,m}/f_0')' (N_{1,n}/f_0')'\, d\eta = C_n \delta_{mn}, \tag{2.13}$$

which indicates that the $N_{1,k}$ functions are orthogonal in the sense of (2.13) and that the $A_{1,k}$ constants may be determined according to the equation

$$A_{1,k} = C_k^{-1} \int_0^\infty (f_0'^4/f_0'') \left[\int_0^\eta \tilde{F}_0\, d\eta / f_0'\right]' (N_{1,k}/f_0')'\, d\eta. \tag{2.14}$$

An additional property of the $N_{1,k}$ eigenfunctions may be obtained from (2.11); multiplication by $H_{1,k} d\eta$, integration from 0 to ∞, and integration by parts lead to

$$\int_0^\infty (f_0'^3/f_0'') H_{1,k}'^2\, d\eta + \int_0^\infty (f_0 f_0''/f_0') H_{1,k}^2\, d\eta = \lambda_{1,k} \int_0^\infty (f_0'^4/f_0'') H_{1,k}^2\, d\eta. \tag{2.15}$$

Since f_0, f_0' and f_0'' are everywhere positive, (2.15) implies that $\lambda_{1,k} > 0$ and thus that the numerical procedures employed for the determination of $\lambda_{1,k}$ would be unsuccessful if negative values of $\lambda_{1,k}$ were sought. Similar considerations involving the complex conjugate of $H_{1,k}$ show that $\lambda_{1,k}$ must be real.

These properties of the $N_{1,k}$ functions imply that they form a complete, orthogonal set with respect to functions having exponential decay at infinity; it is, therefore, possible to improve systematically the solution of the initial value problem under discussion, i.e. to obtain $f_{1,2}(\tilde{s},\eta)$, $f_{1,3}(\tilde{s},\eta)$, etc. This may be seen as follows: If the approximation attendant on $f_{1,i+1} \ll f_{1,i}$ is considered, then (2.1) with $C \equiv 1$ yields for $f_{1,i}$, $i > 1$, the equation

$$f_{1,i\eta\eta\eta} + f_0 f_{1,i\eta\eta} + f_0'' f_{1,i} - 2\tilde{s}(f_0' f_{1,i\tilde{s}\eta} - f_0'' f_{1,i\tilde{s}}) = H_i, \tag{2.16}$$

where $H_i = H_i(\tilde{s},\eta)$ is a given function of the previously obtained functions, $f_{1,i-1}(\tilde{s},\eta)$, etc. With the boundary and initial conditions on f_0 and $f_{1,1}$ selected as above, the corresponding conditions on $f_{1,i}$, $i > 1$ are homogeneous. The requisite solution of (2.16) can be found in terms of a Green's function $G(\tilde{s}, \eta, \tilde{s}_0, \eta_0)$ defined by

$$G_{\eta\eta\eta} + f_0 G_{\eta\eta} + f_0'' G - 2\tilde{s}(f_0' G_{\tilde{s}\eta} - f_0'' G_{\tilde{s}}) = \delta(\eta-\eta_0)\,\delta(\tilde{s}-\tilde{s}_0). \tag{2.17}$$

Now represent G as

$$G(\tilde{s}, \eta, \tilde{s}_0, \eta_0) = \sum_{k=1}^{\infty} G_k(\tilde{s}, \tilde{s}_0, \eta_0)\, N_{1,k}(\eta), \tag{2.18}$$

and $\delta(\eta-\eta_0)$ as

$$\delta(\eta-\eta_0) = \sum_{k=1}^{\infty} D_k(\eta_0)\,(N_{1,k}/f_0')' f_0'^2, \tag{2.19}$$

where from (2.13)

$$D_k = C_k^{-1}(f_0'^2/f_0'')\,(N_{1,k}/f_0')'|_{\eta=\eta_0}. \tag{2.20}$$

Thus the D_k coefficients with η_0 as a parameter are known. Substitution of (2.18) and (2.19) into (2.17), collection of terms in k, and consideration of the equation satisfied by $N_{1,k}$, namely (2.7), lead to

$$2\tilde{s}\frac{dG_k}{d\tilde{s}} + \lambda_{1,k} G_k = -D_k\,\delta(\tilde{s}-\tilde{s}_0). \tag{2.21}$$

For $\tilde{s}_i \leqslant \tilde{s} < \tilde{s}_0$, take $G_k \equiv 0$; the appropriate solution of (2.21) for $\tilde{s}_i \leqslant \tilde{s}_0 < \tilde{s}$ is

$$G_k(\tilde{s}, \tilde{s}_0, \eta_0) = -(D_k/2\tilde{s}_0)\,(\tilde{s}/\tilde{s}_0)^{-\frac{1}{2}\lambda_{1,k}}. \tag{2.22}$$

Thus (2.18), (2.20) and (2.22) define the appropriate Green's function.

The solution for $f_{1,i}(\tilde{s}, \eta)$, $i > 1$, satisfying the initial and boundary conditions is

$$f_{1,i}(\tilde{s}, \eta) = \int_0^{\infty}\int_{\tilde{s}_i}^{\tilde{s}} G(\tilde{s}, \eta, \tilde{s}_0, \eta_0)\, H_i(\tilde{s}_0, \eta_0)\, d\tilde{s}_0\, d\eta_0. \tag{2.23}$$

Thus the higher approximations to the solutions of the initial-value problem can be obtained by quadrature. Generally, a specific quantity such as skin-friction is of interest to that only $f_{1,i\eta\eta}(\tilde{s}, 0)$ must be computed; then from (2.18) with $N''_{1,k}(0) = 1$,

$$f_{1,i\eta\eta}(\tilde{s}, 0) = \int_0^{\infty}\int_{\tilde{s}_i}^{\tilde{s}} \left[\sum_{k=1}^{\infty} G_k(\tilde{s}, \tilde{s}_0, \eta_0)\right] H_i(\tilde{s}_0, \eta_0)\, d\tilde{s}_0\, d\eta_0. \tag{2.24}$$

The solution given by (2.23) is generally applicable to obtaining higher-order approximations for solutions which represent perturbations about the Blasius solution wherein the boundary and initial conditions are satisfied by the combination of the Blasius and the first-order solution.

The first ten eigenvalues and related eigenfunctions are given in figure 1.† It is of interest to note that, to the accuracy available with the computer, the lowest eigenvalue was obtained numerically as 2. In accord with the present findings, Stewartson (1957) showed that one value of λ_1 'and probably the lowest is 2 and that it is unlikely that any of the others are even integers'.

The values of the normalizing factor C_k are given in table 1.

† A tabulation of eigenfunctions is presented in PIBAL Report no. 752, dated August 1962 and in an Addendum thereto dated March 1963. In the evaluation of $A_{1,k}$ for a general function $\tilde{F}_0$, it is convenient to have the function $K_{1,k} \equiv (f_0'^4/f_0'')(N_{1,k}/f_0')'$ tabulated for equal increments in η; this has been done in the aforementioned Addendum.

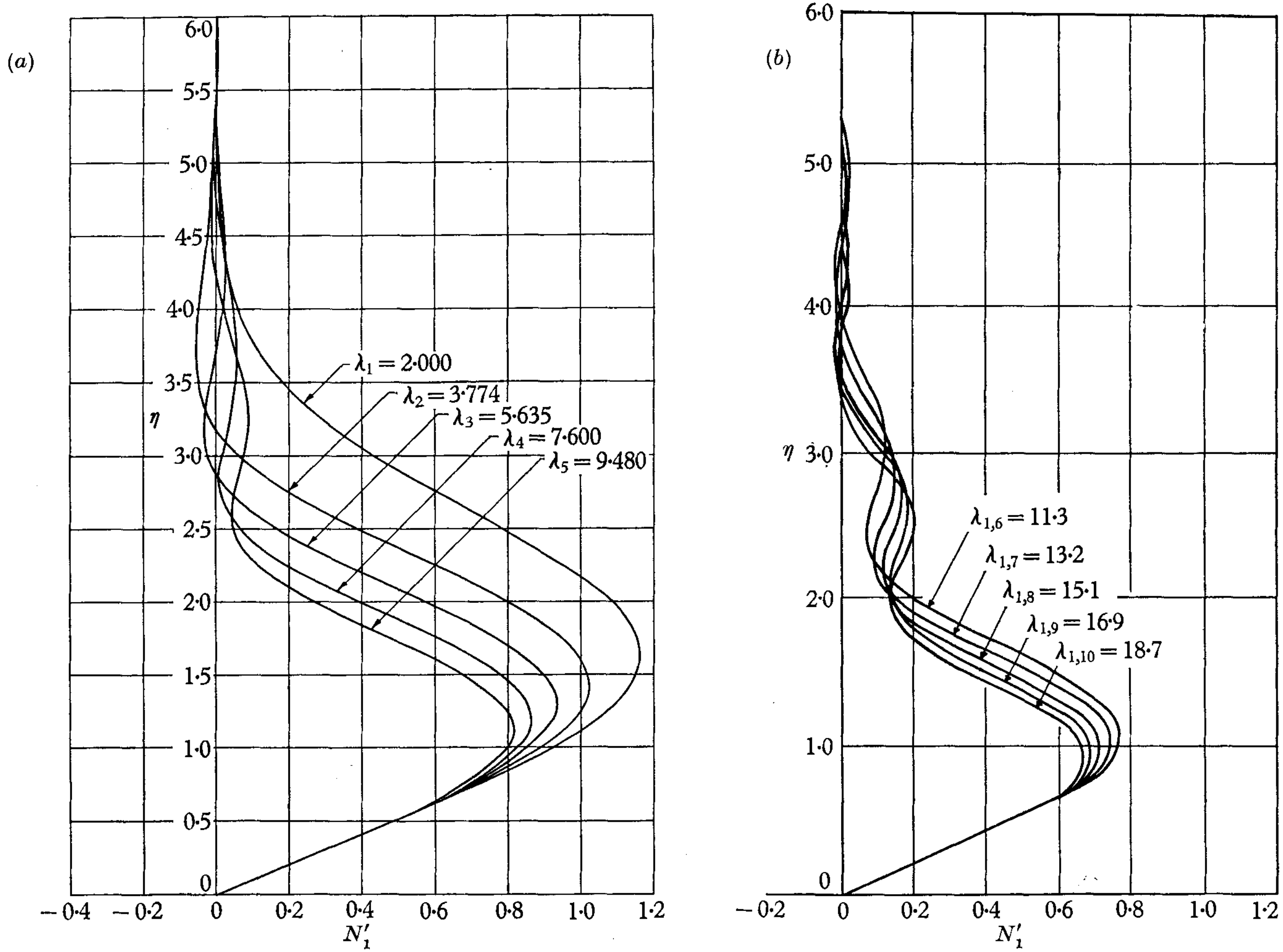

FIGURE 1. (*a*) Eigenfunctions and eigenvalues: initial-value problem, $\lambda_{1,1}$–$\lambda_{1,5}$. (*b*) Eigenfunctions and eigenvalues: initial-value problem, $\lambda_{1,6}$–$\lambda_{1,10}$.

As a concluding remark concerning the above analysis, it is perhaps worth noting that in an application to a particular problem three approximations arise. The first pertains to the representation of the initial profile with a finite number of eigenfunctions; the second approximation is related to the first and pertains to the representation of the Green's function by a finite number of eigenfunctions. The final approximation pertains to the truncation of the expansion of the solution for $f \simeq f_0 + f_{1,1} + \ldots$ after a certain number of terms. The degree of approximation attendant with representation of the initial profile is easily assessed by simple comparison of successive values of the coefficients $A_{1,k}$; however, evaluation of

k	λ_{1k}	C_k	k	λ_{1k}	C_k
1	2·000	2·2672	6	11·3	0·382
2	3·774	1·1447	7	13·2	0·345
3	5·635	0·7813	8	15·1	0·338
4	7·600	0·5065	9	16·9	0·311
5	9·480	0·3095	10	18·7	0·332

TABLE 1

the approximation relative to the Green's function requires recomputation of the solution given by (2.23) with additional eigenfunctions in (2.18) and (2.19). A similar consideration applies to the overall accuracy with respect to the expansion of f. It is noted that in some attempted applications of the present analysis it was found that the first ten eigenfunctions did not provide a satisfactory representation of the initial profile so that calculations of even first-order effects in these cases could not be carried out.

Consider a second perturbation problem which arises when the product of density and viscosity coefficient varies slightly from unity as, for example, from chemical reaction or from a variable wall temperature. Suppose, for example, that

$$C \simeq (h/h_e)^n \quad (n \ll 1). \tag{2.25}$$

Then

$$C \simeq 1 + n\ln(h/h_e) + \ldots$$
$$\simeq 1 + nC_1 + \ldots,$$

where, in general, $C_1 = C_1(\tilde{s}, \eta)$. Suppose further that C_1 is such that for an arbitrarily small but finite range of $\tilde{s}$, namely for $0 \leqslant \tilde{s} \leqslant \tilde{s}_i$, C_1 is a function only of η denoted as $C_{1,i}$ but that for $\tilde{s} > \tilde{s}_i$, $C_1 = C_{1,i}(\eta) + C_{1,d}(\tilde{s}, \eta)$, where $C_{1,d}(\tilde{s}_i, \eta) \cong 0$. For most problems of practical interest such a description of C_1 would appear to be valid.

Now let

$$f(\tilde{s}, \eta) \simeq f_0(\eta) + nf_2(\tilde{s}, \eta) + \ldots, \tag{2.26}$$

so that f_2 will be obtained from (2.5) as

$$f_{2\eta\eta\eta} + f_0 f_{2\eta\eta} + f_0'' f_2 - 2\tilde{s}(f_0' f_{2\tilde{s}\eta} - f_0'' f_{2\tilde{s}}) = -(C_1 f_0'')_\eta. \tag{2.27}$$

In accordance with the treatment of C_1 let

$$f_2(\tilde{s}, \eta) = f_{2,i}(\eta), \tag{2.28}$$

for $0 \leqslant \tilde{s} \leqslant \tilde{s}_i$ and

$$f_2(\tilde{s}, \eta) = f_{2,i}(\eta) + f_{2,d}(\tilde{s}, \eta), \tag{2.29}$$

for $\tilde{s} > \tilde{s}_i$; then from (2.27)

$$f'''_{2,i}+f_0 f''_{2,i}+f''_0 f_{2,i} = -(C_{1,i} f''_0)', \tag{2.30}$$

while

$$(f_{2,d})_{\eta\eta\eta}+f_0(f_{2,d})_{\eta\eta}+f''_0 f_{2,d}-2\tilde{s}[f'_0(f_{2,d})_{\tilde{s}\eta}-f''_0(f_{2,d})_{\tilde{s}}] = -(C_{1,d} f''_0)_\eta. \tag{2.31}$$

Clearly the solution for $f_{2,i}$ applies for all $\tilde{s}$ in the special case of C_1 depending only on η. The boundary conditions for both parts of f_2 are homogeneous and the initial conditions on $f_{2,d}$ at $\tilde{s} = \tilde{s}_i$ are $f_{2,d}(\tilde{s}_i, \eta) \equiv 0$.

With $C_{1,i}$ specified as a function of η, (2.30) can be integrated numerically from $\eta = 0$. Only two computer runs are necessary, one with the right-hand side equal to zero and with $f_{2,i}(0) = f'_{2,i}(0) = 0, f''_{2,i}(0) = 1$, and the second with the same conditions at $\eta = 0$ but with the right-hand side present. Because of the behaviour of the left-hand side of (2.30) as $\eta \to \infty$, any arbitrary multiple of the first solution plus the second can be found so that $f'_{2,i}(\infty) = 0$.

A special case of this solution prevails if both (2.25) and the Crocco relation $g = g_{w,0}+(1-g_{w,0})f_\eta$ apply. Then

$$C_{1,i} = \ln\{[g_{w,0}+(1-g_{w,0})f'_0-\tilde{m}f'^2_0](1-\tilde{m})^{-1}\}. \tag{2.32}$$

The right-hand side of (2.30) in this case has as parameters $g_{w,0}$ and $\tilde{m}$; it is thus convenient to generate three independent, complementary solutions which permit the solution for an arbitrary right-hand side to be determined by quadrature, following the method of variation of parameters. Identify these three solutions as $f_2^{(k)}$, where

$$\left.\begin{aligned} &f_2^{(1)}(0) = 1, \quad f_2^{(1)\prime}(0) = f_2^{(1)\prime}(\infty) = 0, \\ &f_2^{(2)}(0) = 0, \quad f_2^{(2)\prime}(0) = 1, \quad f_2^{(2)\prime}(\infty) = 1, \\ &\quad f_2^{(3)} = f'_0/f''_{0w}. \end{aligned}\right\} \tag{2.33}$$

The solutions $f_2^{(1)}$ and $f_2^{(2)}$ have been found numerically and are shown in figure 2.† In the numerical analysis for the determination of $f_2^{(k)}$, $k = 1, 2$, there is sought the value of $f_2^{(k)\prime\prime}(0)$, which yields the proper asymptotic behaviour for $f_2^{(k)}$. Note that (2.8) is the asymptotic solution for (2.30) with a zero right-hand side provided λ_1 is set equal to zero and N'_1 replaced by $f_2^{(k)\prime}$; thus, $f_2^{(1)\prime\prime}(0)$ and $f_2^{(2)\prime\prime}(0)$ are selected so that $\alpha_2^{(1)} = 0$ and $\alpha_2^{(2)} = 1$, respectively. Clearly, the values of $f_2^{(k)\prime\prime}(0)$ can be obtained by combining linearly two numerical solutions.

The solution for $f_{2,d}(\tilde{s}, \eta)$ is identical to that for (2.16), i.e. it is given by (2.23) with $H_i = -(C_{1,d} f''_0)_\eta$. Thus a first-order effect on the velocity field due to alteration of the product of density and viscosity can be readily obtained by application of the eigenfunctions presented here and by numerical quadrature.

As a third perturbation consider a flow involving mass transfer at the wall of such strength and distribution that the boundary layer is only slightly altered from that given by the Blasius solution. Now, in general,

$$\rho v r^j = -(d\tilde{s}/dx)((2\tilde{s})^{\frac{1}{2}}f)_{\tilde{s}}+(2\tilde{s})^{\frac{1}{2}}(\partial\eta/\partial x)f_\eta, \tag{2.34}$$

or

$$(\rho v)_w/\rho_e u_e = -\mu_e r^j((2\tilde{s})^{\frac{1}{2}}f_w)_{\tilde{s}}. \tag{2.35}$$

† These solutions are tabulated in PIBAL Report No. 752, dated August 1962.

Several types of perturbations could now be considered in terms of (2.34); e.g. if the mass transfer is distributed so that f_w = const., then the perturbed flow will be similar, i.e. will be described by an ordinary differential equation in η. A second type of perturbation will be considered herein; namely, let the mass transfer in the region of injection be uniform, i.e. $(\rho v)_w$ = const. and let the radius, if finite,

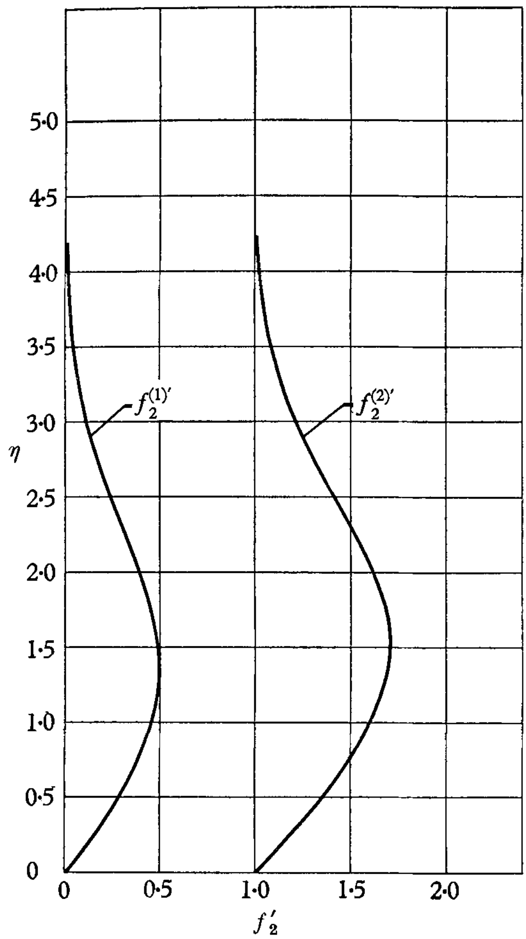

FIGURE 2. Perturbation velocity profiles: variable mass-density—viscosity ratio. $f_{2w}^{(1)''} = 0{\cdot}7235$; $f_{2w}^{(2)''} = 0{\cdot}7044$.

be a linear function of x, i.e. $r = Bx$ or $r \sim \tilde{s}^{\frac{1}{3}}$. It is convenient to introduce the parameter

$$\Phi \equiv \left[\frac{(\rho v)_w}{\rho_e u_e \mu_e}\right]\left[\frac{\rho_e u_e \mu_e}{B(2j+1)}\right]^{j/(2j+1)}, \tag{2.36}$$

so that (2.35) becomes

$$\frac{d}{d\tilde{s}}[(2\tilde{s})^{\frac{1}{2}} f_w] = -\Phi \tilde{s}^{-j/(2j+1)}, \tag{2.37}$$

or upon integration

$$f_w = -\Phi(2\tilde{s})^{-\frac{1}{2}}\left(\frac{2j+1}{k+1}\right)[\tilde{s}^{(j+1)/(2j+1)} - \tilde{s}_i^{(j+1)/(2j+1)}]. \tag{2.38}$$

In (2.38) it has been assumed that injection starts at $\tilde{s} = \tilde{s}_i$; thus, mass transfer is considered to be initiated downstream of the leading edge. However, note that no difficulty arises if $\tilde{s}_i = 0$. The perturbation of the flow will be related to the value of f_w; thus, for the analysis to be valid, $|f_w| \ll 1$. As may be seen from (2.38) this inequality can be satisfied by appropriate combinations of rate and extent of mass transfer, i.e. of $(\rho v)_w$ and $\tilde{s}$.

If a perturbation parameter $\epsilon_3 \equiv (\rho v)_w/(\rho_e u_e)$ is introduced, and if

$$f(\tilde{s},\eta) \simeq f_0(\eta) + \epsilon_3 f_3(\tilde{s},\eta), \tag{2.39}$$

then linearization and separation of variables lead as before to $S_3 \sim \tilde{s}^{\frac{1}{2}\lambda_3}$ and to

$$N_3''' + f_0 N_3'' - \lambda_3 f_0' N_3' + f_0''(1+\lambda_3) N_3 = 0. \tag{2.40}$$

The boundary conditions for (2.40) can be conveniently taken to be

$$N_3(0) = 1, \quad N_3'(0) = N_3'(\infty) = 0. \tag{2.41}$$

From (2.38) it is clear that the boundary conditions on $f_3(\tilde{s},\eta)$ can be satisfied if solutions to (2.40) are available with $\lambda_3 = \pm 1$ for two-dimensional flows ($j = 0$) and with $\lambda_3 = \frac{1}{3}$ or -1 for axisymmetric flows ($j = 1$). The solutions for $\lambda_3 = 1, \frac{1}{3}$ have been obtained numerically and are given in figure 3.† The solution for $\lambda_3 = -1$ is exactly $N_3 \equiv 1$. It is noted that the asymptotic solution of (2.40) is (2.8) with λ_1 and N_1' replaced by $-\lambda_3$ and N_3', respectively. The requirement of either boundedness or exponential behaviour as $\eta \to \infty$, depending on the sign of λ_3, necessitates $\alpha_2 = 0$; thus, the numerical analysis requires $N_3''(0)$ to be selected so that $\alpha_2 = 0$.

It is noted that

$$f(\tilde{s},\eta) = f_0(\eta) + \frac{(\rho v)_w}{\rho_e u_e} \sum_{k=1}^{2} A_{3,k} \tilde{s}^{\frac{1}{2}\lambda_{3,k}} N_{3,k}, \tag{2.42}$$

where the coefficients $A_{3,k}$ can be determined by comparing (2.42) with (2.38). It should also be noted that $f_\eta(\tilde{s}_i,\eta) \neq f_0$, if $\tilde{s}_i \neq 0$; it is therefore necessary to add to this solution a perturbation solution of the type discussed above in order to make the initial profile correspond to the Blasius solution or to a prescribed deviation from it.

Again for this perturbation higher-order approximations for the solution for $f(\tilde{s},\eta)$ (cf. (2.39)) can be obtained in the same manner as above. Thus, for example, denote the solution for f_3 in (2.42) by $f_{3,1}$ and the next order solution by $f_{3,2}$. Then the equation for $f_{3,2}$ is formally identical to (2.16) where H_i depends on $f_{3,1}$; the solution for $f_{3,2}$ is (2.23).

3. Applications

In this section are presented several typical applications of the perturbation solutions obtained in the previous section; these have been selected, in general, so that comparison with other, more accurate analyses is possible.

Howe (1959) and Pallone (1961) have considered the flow over a two-dimensional, permeable wall followed at $x = L$ by an impermeable surface. The mass transfer on the porous surface is distributed so that the boundary layer thereon is similar and described, for example, by the analysis of Low (1955). The

† The solutions are tabulated in PIBAL Report no. 752, dated August 1962.

boundary-layer properties on the impermeable surface are difficult to obtain accurately from a simple integral method; this is demonstrated by Howe (1959) who gives a comparison between exact numerical calculations and the results of several analyses based on the integral method. Of interest herein will be the variation of skin friction on the impermeable surface normalized with respect to that which

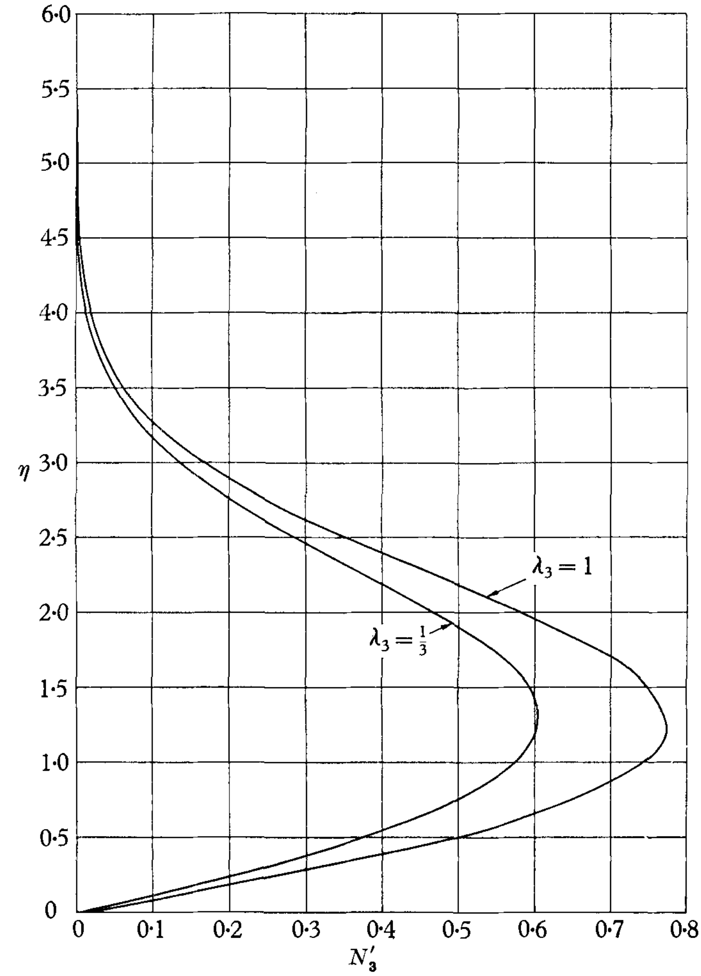

FIGURE 3. Perturbation velocity profiles: suction-injection problem. $f''(0) = 1{\cdot}2274$ for $\lambda_3 = 1{\cdot}0$; $f''(0) = 0{\cdot}90387$ for $\lambda_3 = \frac{1}{3}$.

would prevail at the same station $x > L$ if no upstream mass transfer exists. This problem may be treated as an initial value problem as follows: The coefficients $A_{1,k}$ in (2.9) may be selected so that the similar solution applicable on the surface $x < L$ is matched, and the resulting solution employed to find the distribution of $c_f/c_{f,0}$ with x/L. Note that $c_f = 2^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}f''(0)\ (\rho_e u_e x/\mu_e)^{-\frac{1}{2}}$ provided $\rho\mu = \rho_e\mu_e$ so that

$$c_f/c_{f,0} = f''(0)/f_0''(0) \simeq 1 + [f_0''(0)]^{-1} \sum_{k=1}^{N} A_{1,k}(x/L)^{-\frac{1}{2}\lambda_{1,k}}, \tag{3.1}$$

for both $j = 0$ and $j = 1$ where only the first-order perturbation is indicated.

In figure 4 the velocity profile obtained by this procedure with ten $A_{1,k}$ coefficients computed from (2.14) is compared with the actual initial profile for $f_w = -0{\cdot}5$. In figure 5 the distribution of $c_f/c_{f,0}$ with x/L given by (3.1) is com-

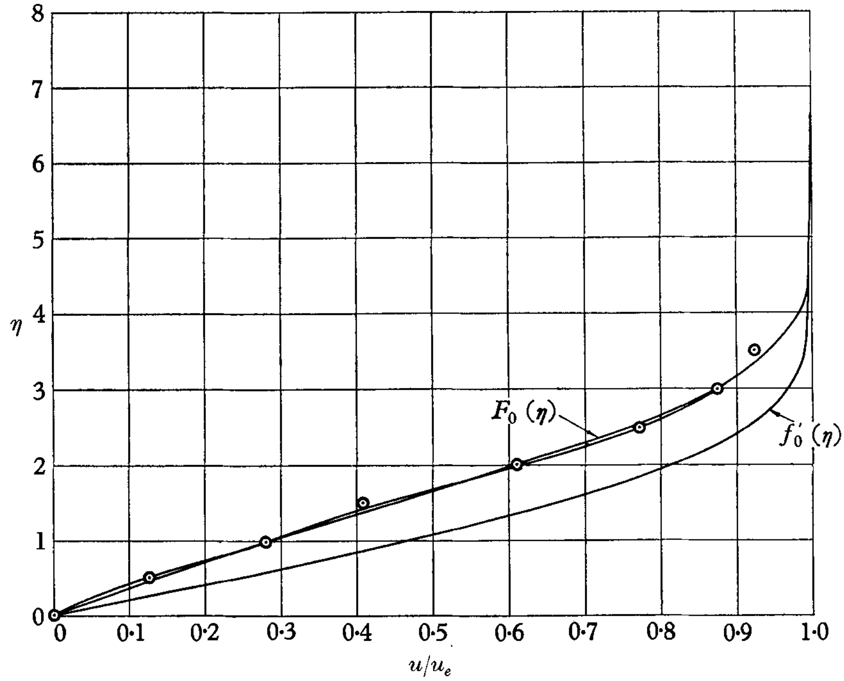

FIGURE 4. Comparison of initial profile. —⊙—, Fitted profile.

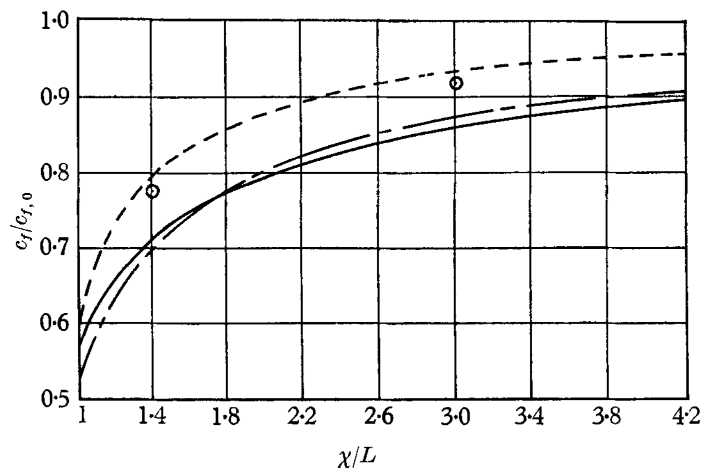

FIGURE 5. Distribution of skin friction. – – –, Pallone; ——, addendum; — — —, PIBAL 752; ⊙, with $f_2(\bar{s}\eta)$.

pared with that given by Rubesin & Inouye (1957) and Pallone (1961). Reasonable agreement is noted. This agreement is improved (cf. figure 5) by consideration of the second-order perturbation which has been computed by application of (2.24) to two values of x/L, namely $x/L = 1{\cdot}4$ and $3{\cdot}0$.

The results of the analysis for variable $\rho\mu$ can be compared to the exact calculations which are due to van Driest (1952) and which were carried out for constant coefficients of specific heat, for the Sutherland viscosity-temperature relation and for a constant wall temperature. In this case only the $f_{2,i}(\eta)$ solution is necessary and

$$g_{w.0} = (1-\tilde{m})(T_w/T_\infty),$$

$$\tilde{m} = [1+(2/\gamma-1)M_\infty^{-2}]^{-1}.$$

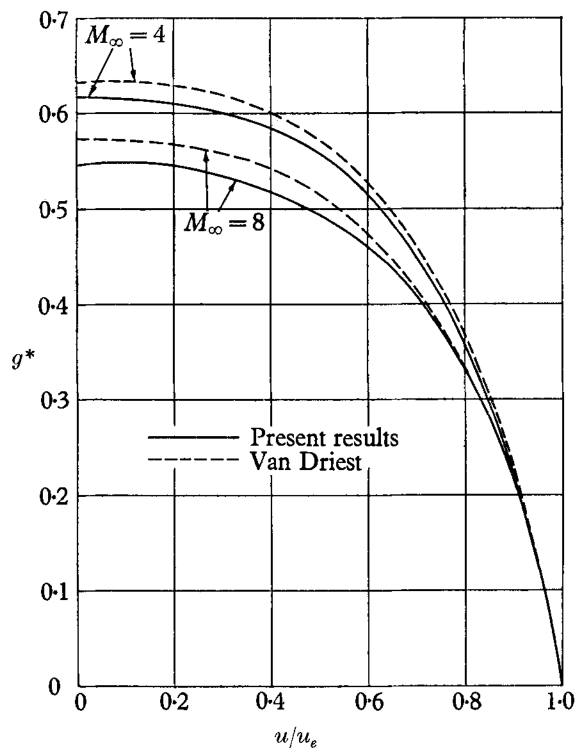

FIGURE 6. Distribution of shear function. $T_w/T_\infty \simeq 1{\cdot}00$; $n = -0{\cdot}30$.

For comparison it is convenient to consider the shear parameter g^* and the parameter $C_f R_{ex}^{\frac{1}{2}}$ of van Driest; it is easy to show that

$$g^* = C\sqrt{2}f'',$$

and that
$$C_f R_{ex}^{\frac{1}{2}} = 2\sqrt{2}C(0)f''(0).$$

The accuracy of the velocity profiles given by the perturbation solution can easily be shown in terms of g^* as a function of f'. In figure 6 this comparison is shown for $M_\infty = 4$ and for $T_w/T_\infty = 1$. A value of $n = -0{\cdot}3$ has been assumed. Satisfactory agreement is seen.

The comparison for skin friction for a range of wall temperature ratios and free stream Mach numbers is shown in table 2. Again satisfactory agreement is noted.

The perturbations due to mass transfer have been compared to the finite difference calculation due to Smith & Clutter (1961) for the two-dimensional, incompressible flow over a flat plate with uniform suction. For this case (2.42) leads to

$$f(\bar{s},\eta) = f_0 - \left[\frac{(\rho v)_w}{\rho_e u_e \mu_e}\right] N_{3,1}(\bar{s}/2)^{\frac{1}{2}}. \tag{3.2}$$

T_w/T_∞	M_∞	Present report	Van Driest
1·04	4·0	1·234	1·27
4·17	4·0	1·092	1·14
1·00	8·0	1·171	1·15
2·00	8·0	1·164	1·11

TABLE 2. Comparison of skin friction—$C_f R_{ex}^{\frac{1}{2}}$.

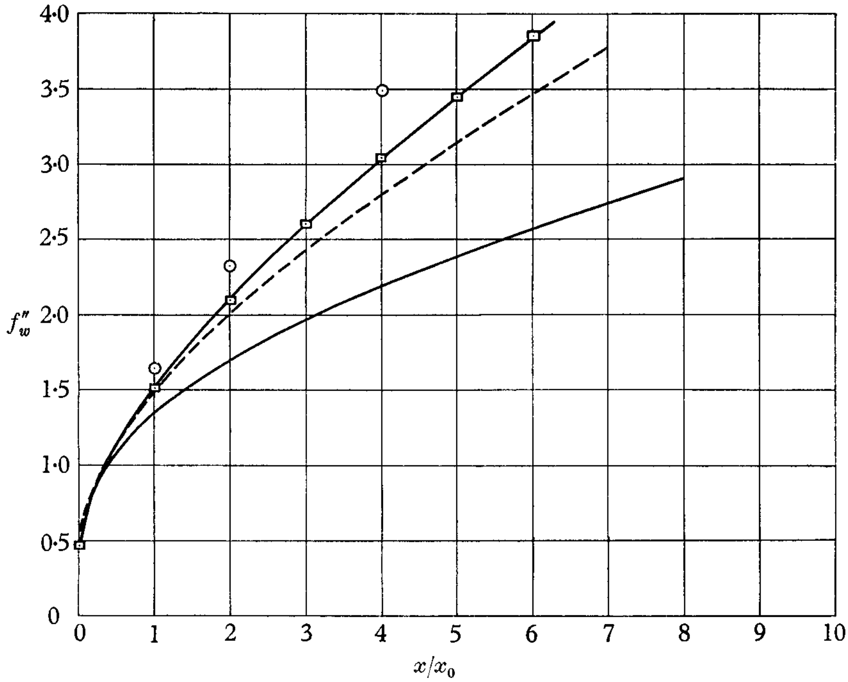

FIGURE 7. Distribution of shear parameter: uniform suction. $-(v_w/u_e)(u_e x_0/\nu_e)^{\frac{1}{2}} = 1$; $x_0 = 1$. ——, First order; ⊙, second order with five eigenfunctions; —□—, second order with ten eigenfunctions; – –, Smith & Clutter.

The wall shear parameter obtained from (3.2) is compared to the results of Smith & Clutter in figure 7. The next order term, i.e. $\epsilon_3^2 f_{3,2}$, is obtained from an equation of the form (2.16) with

$$H_i = (s/2\mu_e^2)(N'^2_{3,1} - 2N''_{3,1}N_{3,1}). \tag{3.3}$$

It is interesting to note that (3.2) and (3.3) imply that for this case a convenient expansion would have typical term $[(\epsilon_3/\mu_e)(\bar{s}/2)^{\frac{1}{2}}]^n N_{3,n}(\eta)$. The results for five and ten eigenvalues in the representation of the Green's function are shown in figure 7. The significant improvement in accuracy by the addition of the $f_{3,2}$ function may be noted.

4. Concluding remarks

Flows which are described by perturbations about the Blasius solution have been considered. These perturbations are related to three problems: (1) initial velocity profiles deviating from the Blasius solution, (2) flows with a variable product of mass-density and viscosity, and (3) flows with mass transfer at the wall. Solution of the first problem leads to a set of eigenfunctions which permit higher-order approximations to each problem to be obtained in a systematic manner by means of quadrature.

Several comparisons of the results of the analysis with more accurate calculations have been carried out; good agreement has been obtained.

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Appendix

Considered herein are the procedures used for the selection of the eigenvalues $\lambda_{1,k}$. For completeness (2.7) is repeated; under consideration is the equation

$$N'''_{1,k}+f_0 N''_{1,k}+\lambda_{1,k} f'_0 N'_{1,k}+(1-\lambda_{1,k})f''_0 N_{1,k}=0, \qquad \text{(A 1)}$$

subject to the conditions $N_{1,k}(0)=N'_{1,k}(0)=N'_{1,k}(\infty)=0$. The most accurate and sensitive method for the selection of those values of $\lambda_{1,k}$ yielding exponential decay as $\eta\to\infty$ appears to be the following: consider values of η sufficiently large so that f_0, f'_0, and f''_0 take on their asymptotic values; then (A 1) becomes

$$N'''_{1,k}+(\eta-\kappa)N''_{1,k}+\lambda_{1,k}N'_{1,k}\simeq 0. \qquad \text{(A 2)}$$

Let $N'_{1,k}=e^{-\frac{1}{4}\tilde{\eta}^2}Z_k(\tilde{\eta})$, where $\tilde{\eta}\equiv\eta-\kappa$ so that (A 2) becomes

$$\left(\frac{d^2Z_k}{d\tilde{\eta}^2}\right)+(\lambda_{1,k}-\tfrac{1}{4}\tilde{\eta}^2)Z_k=0. \qquad \text{(A 3)}$$

This is Weber's equation [see, for example, Whittaker & Watson (1958)]. From the theory of this equation it is known to have an oscillating solution for $\eta<2\lambda^{\frac{1}{2}}_{1,k}$ and to have an exponential behaviour for $\eta>2\lambda^{\frac{1}{2}}_{1,k}$. In particular, in the former region the solution corresponding to a particular eigenfunction will have one zero fewer than the next higher eigenfunction. It is to be noted that one solution to (A 3) is proportional to $e^{\frac{1}{4}\tilde{\eta}^2}$; it is this solution which leads to the power law behaviour of $N'_{1,k}$. This suggests the following procedure for the selection of the eigenvalues: With a particular $\lambda_{1,k}$ assumed the integration from $\eta=0$ is carried out to $\eta=\eta^*$, a value of roughly 5. With the values of $N'_{1,k}$ and $N''_{1,k}$ at $\eta=\eta^*$ initial conditions for the integration of (A 3) may be computed and the numerical integration then carried out in terms of Z_k. All solutions will diverge for some $\eta>2\lambda^{\frac{1}{2}}_{1,k}$; however, for some $\lambda_{1,k}$'s, $Z_k\to\infty$ as $\tilde{\eta}\to\infty$, while for others $Z_k\to-\infty$

as $\tilde{\eta} \to \infty$. It will be recognized that the introduction of the exponential in the reduction of (A 2) to the second order (A 3) effectively exposes the power-law contribution to $N'_{1,k}$ and thus permits accurate evaluation of those values of $\lambda_{1,k}$ yielding only exponential behaviour. Unfortunately there are no simple solutions to (A 3) so that the procedure above is somewhat cumbersome.

An alternative approach is as follows: If (2.11) is written with the asymptotic approximations for f_0, f'_0 and f''_0 considered, there results

$$H''_{1,k} + (\eta - \kappa) H'_{1,k} + \lambda_{1,k} H_{1,k} = 0. \tag{A 4}$$

The approximate solution of (A 4) valid for $|1-\lambda_{1,k}|(\eta-\kappa)^{-2} \ll 1$ is well known in boundary-layer theory (cf. Cohen & Reshotko 1956) and is

$$H_{1,k} \simeq \alpha_1(\eta-\kappa)^{-(1-\lambda_{1,k})} \exp[-(\eta-\kappa)^2/2] + \alpha_2(\eta-\kappa)^{-\lambda_{1,k}}, \tag{A 5}$$

where α_1 and α_2 are integration constants. Now N'_1 can be obtained from (A 5) as

$$N'_1 \simeq \alpha_1(\eta-\kappa)^{-(1-\lambda_1)} \exp[-\tfrac{1}{2}(\eta-\kappa)^2] + \alpha_2(\eta-\kappa)^{-\lambda_1} + \alpha_3 \exp[-\tfrac{1}{2}(\eta-\kappa)^2]. \tag{A 6}$$

Consider now that a numerical solution of (A 1) has been carried out for a particular value of $\lambda_{1,k}$ to $\eta = \eta^*$, large in the sense discussed above. Then α_1, α_2 and α_3 in (A 6) can be selected so that the asymptotic solution valid for $\eta > \eta^*$ is continuous with the numerical solution. In particular, the arbitrary constant α_2 can be expressed as

$$\alpha_2 \simeq \{N''_1 + (\eta-\kappa) N'_1[1 + (1-\lambda_1)(\eta-\kappa)^{-2}]\}_{\eta=\eta^*}, \tag{A 7}$$

where the same approximation is made here as employed in (A 5). By determining the behaviour of α_2 versus $\lambda_{1,k}$ the values of the latter resulting in $\alpha_2 = 0$ can be determined.

This second procedure was used in determining the first five eigenvalues presented here; it fails, however, for the higher eigenfunctions because, with the increasing η^* necessary for (A 6) to apply, (A 7) involves small differences and α_2 cannot be determined therefrom. The second five eigenfunctions were found in an approximate manner as follows: Stewartson (1957) analysed (A 1) for large values of $\lambda_{1,k}$ ($\lambda_{1,k} \gg 1$) and arrived at the following approximate expression for the eigenvalues:

$$\lambda_{1,s} \simeq 2s + 0{\cdot}27(2s)^{\frac{1}{2}} + 1{\cdot}87,$$

where s is an integer identifying the eigenvalue. The 'exact' values of $\lambda_{1,k}$, $k = 1, \ldots 5$, found here are found to be correlated by the equation

$$\lambda_{1,k} \cong 0{\cdot}891\{2(k-1) + 0{\cdot}27[2(k-1)]^{\frac{1}{2}} + 1{\cdot}87\}, \tag{A 8}$$

which is in good agreement with Stewartson's expression. Now (A 8) was employed to obtain the eigenvalues for $k > 5$. The resulting eigenfunctions have the proper oscillatory behaviour for $\eta < 2\lambda^{\frac{1}{2}}_{1,k}$. In addition, for values of $\lambda_{1,k}$ different from those given by (A 8) by $\pm 0{\cdot}1$ the eigenfunctions in terms of $N'_{1,k}$ are not significantly altered except for $\eta \gtrsim 2\lambda^{\frac{1}{2}}_{1,k}$. Accordingly, these approximate eigenfunctions are believed to be sufficiently accurate for most purposes.

The numerical integrations involved in the perturbation solutions were carried out on an IBM 650 Data Processing System using a standard Kutta–Runge program with self-selecting step-sizes. Computing time was approximately 15 min per run. The Blasius function which was input was obtained by integration for $0 < \eta \leqslant 6$ and from its asymptotic representation for $\eta \geqslant 6$. After final determination of the eigenvalues, the eigenfunctions and various functions related to them were tabulated for equal increments of η on a Bendix G-15 computer.

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